Stochastic differential equations with non-negativity constraints driven by fractional Brownian motion

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Abstract

In this paper we consider stochastic differential equations with non-negativity constraints, driven by a fractional Brownian motion with Hurst parameter $H > \mathbf{1}/2$. We first study an ordinary integral equation, where the integral is defined in the Young sense, and we prove an existence result and the boundedness of the solutions. Then we apply this result pathwise to solve the stochastic problem.

Keywords: stochastic differential equations, normal reflection, fractional Brownian motion, Young integral

AMS 2000 MSC: 60H05, 60H20

Running head: stochastic differential equations with constraints

1 Introduction

The study of differential equations driven by a fractional Brownian motion has been developed in recent years. It has been done, using either the formalism of the rough path analysis [4, 15, 11] or the fractional calculus [17, 20]. As is natural, afterwards has been considered some of the possible generalizations of the diffusion processes. For instance, in the literature we can find now papers about PDEs [18, 3, 5, 12], Volterra equations [6, 7, 2] or systems with delay [10, 9, 16, 14].

Since in some applications the quantities of interest are naturally positive, it is also natural to consider equations with positivity constraints. As far as the authors know, it has only been studied up to now the case of delay equations with positivity constraints [1]. As we shall see, the present paper follows these steps and we shall deal with stochastic equations with positivity constraints driven by a fractional Brownian motion with Hurst parameter H > 1/2. More precisely, we deal with a stochastic differential equation with normal reflection on $\mathbb{R}^d$ of the form:

$$X(t) = x(0) + \int_0^t b(s, X(s))ds + \int_0^t \sigma(s, X(s))dW_s^H + Y(t), \quad t \in (0, T], \tag{1.1}$$

where $x^i(0) > 0$, for $i = 1, \ldots, d$, $W^H = \left\{W^{H,j},\ j = 1, \ldots, m\right\}$ are independent fractional Brownian motions with Hurst parameter $H > \frac{1}{2}$ defined in a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$, while Y, the so-called regulator term, is a vector-valued non-decreasing process which ensures that the non-negativity constraints on X are enforced. This can be obtained in the following customary way:

Set

$$Z(t) = x(0) + \int_0^t b(s, X(s))ds + \int_0^t \sigma(s, X(s))dW_s^H, \qquad t \in [0, T]. \tag{1.2}$$

It is known (see e.g. [8, 13]) that we have an explicit formula for the regulator term Y in terms of Z, the so-called reflector term: for each $i = 1, \ldots, d$

$$Y^i(t) = \max_{s \in [0,t]} \left(Z^i(s)\right)^-, \qquad t \in [0, T].$$

Then the solution of (1.1) satisfies

$$X(t) = Z(t) + Y(t) \qquad t \in [0, T].$$

We call (1.1) a stochastic differential equation with reflection driven by a fractional Brownian motion and to the best of our knowledge this problem has not been considered before in the wide literature on stochastic differential equations.

In order to deal with non-negative constraints we use the Skorohod's mapping. Set

$$\mathcal{C}_+(\mathbb{R}_+, \mathbb{R}^d) := \left\{x \in \mathcal{C}(\mathbb{R}_+, \mathbb{R}^d) : x(0) \in \mathbb{R}_+^d\right\}.$$

Let us recall now the Skorokhod problem.

Definition 1.1 *Given a path $z \in \mathcal{C}_+(\mathbb{R}_+, \mathbb{R}^d)$, we say that a pair (x, y) of functions in $\mathcal{C}_+(\mathbb{R}_+, \mathbb{R}^d)$ solves the Skorokhod problem for z with reflection if*

1. *$x(t) = z(t) + y(t)$ for all $t \geq 0$ and $x(t) \in \mathbb{R}_+^d$ for each $t \geq 0$,*
2. *for each $i = 1, \ldots, d$, $y^i(0) = 0$ and y^i is nondecreasing,*
3. *for each $i = 1, \ldots, d$, $\int_0^t x^i(s)dy^i(s) = 0$ for all $t \geq 0$, so y^i can increase only when x^i is at zero.*

Then we have an explicit formula for y in terms of z: for each $i = 1, \ldots, d$

$$y^i(t) = \max_{s \in [0,t]} \left(z^i(s)\right)^-.$$

The path x is called the reflector of z and the path y is called the regulator of z. We will use the Skorokhod mapping to force a continuous real-valued function to be non-negative by means of reflection at the origin. We will apply

it to each path of Z defined by (1.2). Note that, since we are dealing with a multidimensional case, the mapping will be applied to each component.

At this point we have also to explain how the stochastic integral appearing in our equation has to be understood. Since the Hurst parameter $H > 1/2$, the stochastic integral is defined using a pathwise approach. We shall first consider a variation of the Young integration theory [19] (called algebraic integration, introduced in [11]), in order to define a deterministic integral with respect to Hölder continuous function. Then we will prove our results for deterministic equations and at the end we will easily apply them pathwise to the fractional Brownian motion.

Let us say a few words about the strategy we have followed in order to prove our results. Existence and uniqueness results are usually proved together using a fixed point argument. In order to apply this type of argument, we have to be able to control the difference between two solutions of our system, $\|x^1 - x^2\|$, where $\|\cdot\|$ denotes a generic norm. Dealing with stochastic integrals with respect to fractional Brownian motion a well-posed norm to work with is the λ-Hölder one. However, as can we seen in Remark 3.6, it is not possible to control the difference between two regulator terms y^1 and y^2 using a λ-Hölder norm. So, we are not able to use a fixed point argument. Actually, the existence result will be proved using an equicontinuous argument, while the uniqueness is still an open problem. We are only able to prove the uniqueness result just up to the first time the (up to then) unique solution has the first component being zero.

Here is how our paper is structured: in Section 2, we will state our main results. Then in Section 3 we shall recall the basic notions of the algebraic integration theory, the Young integration and the Skorohod mapping. Section 4 will contain the study of the deterministic integral equations: the existence and boundedness of the solutions. Finally, Section 5 will be devoted to recall how to apply the deterministic results to the stochastic case.

2 Main results

For any $0 < \lambda \leq 1$, denote by $C^\lambda(s,t;\mathbb{R}^d)$ the space of $\lambda-$Hölder continuous functions, namely the functions $f : [s,t] \to \mathbb{R}^d$ such that

$$\|f\|_{\lambda,[s,t]} := \sup_{s \leq u < v \leq t} \frac{|f(v) - f(u)|}{(v-u)^\lambda} < \infty \ ,$$

and

$$\|f\|_{\infty,[s,t]} := \sup_{u \in [s,t]} |f(u)| \, .$$

Let us consider the following assumptions on the coefficients.

(H1) $\sigma : [0,T]\times\mathbb{R}^d \to \mathbb{R}^{d\times m}$ is bounded and there exists a constant K_0 such that the following properties hold:

$$\begin{cases} i) \quad \text{Lipschitz continuity} \\ |\sigma(t,x)-\sigma(t,y)| \le K_0|x-y|, \quad \forall\, x,y\in\mathbb{R}^d,\ \forall\, t\in[0,T] \\ ii) \quad \nu\text{-Hölder continuity in time} \\ |\sigma(t,x)-\sigma(s,x)| \le K_0|t-s|^{\nu}, \quad \forall\, x\in\mathbb{R}^d,\ \forall\, t,s\in[0,T]\,. \end{cases}$$

(H2) $b : [0,T]\times\mathbb{R}^d \to \mathbb{R}^d$ is bounded and Lipschitz continuous in x, that is, there exists a constant K_0 such that

$$|b(t,x)-b(t,y)| \le K_0|x-y|, \quad \forall\, x,y\in\mathbb{R}^d,\ \forall\, t\in[0,T]\,.$$

Under these assumptions we are able to prove that our problem admits a solution. Our main result reads as follows:

Theorem 2.1 *Assume that σ and b satisfy hypothesis* **(H1)** *and* **(H2)***, respectively, with $\nu \ge H$. Set $\lambda_0 \in (\frac{1}{2}, H)$. Then equation (1.1) admits a solution*

$$X \in L^0(\Omega,\mathcal{F},\mathbb{P};C^{\lambda_0}(0,T;\mathbb{R}^d)).$$

It can also be seen that the solution has moments of any order.

Theorem 2.2 *Assume that σ and b satisfy hypothesis* **(H1)** *and* **(H2)***, respectively, with $\nu \ge H$. Set $\lambda_0 \in (\frac{1}{2}, H)$. If X is a solution of (1.1), then*

$$E(\|X\|^p_{\lambda_0,[0,T]}) < \infty, \quad \forall p \ge 1.$$

3 Preliminaries

As mentioned in the introduction, we are concerned with stochastic integral with respect to a fractional Brownian motion with Hurst parameter $H > 1/2$. In order to define the stochastic integral we will use the Young integration. We will follow the algebraic approach introduced in [11] (see also [12, 6, 7]). For the sake of completeness, we will recall some basic facts and notations from those papers. We refer the reader to the same references for a detailed presentation.

In addition, we will recall some known results on the Skorohod mapping and prove an inequality that we will need throughout the paper.

3.1 Increments

Let us begin with the basic algebraic structures which will allow us to define a pathwise integral with respect to irregular functions: first of all, for a real number $T > 0$, a vector space V and an integer $k \ge 1$ we denote by $\mathcal{C}_k(V)$ (or by $\mathcal{C}_k([0,T];V)$) the set of continuous functions $g : [0,T]^k \to V$ such that $g_{t_1\cdots t_k} = 0$ whenever $t_i = t_{i+1}$ for some $i \le k-1$. Such a function will be

called a $(k-1)$-*increment*, and we will set $\mathcal{C}_*(V) = \cup_{k\geq 1}\mathcal{C}_k(V)$. An elementary operator on $\mathcal{C}_k(V)$ is δ, defined as follows:

$$\delta : \mathcal{C}_k(V) \to \mathcal{C}_{k+1}(V), \qquad (\delta g)_{t_1\cdots t_{k+1}} = \sum_{i=1}^{k+1}(-1)^{k-i} g_{t_1\cdots \hat{t}_i \cdots t_{k+1}}, \tag{3.1}$$

where $\hat{t}_i$ means that this argument is omitted. A fundamental property of δ is that $\delta\delta = 0$. Set $\mathcal{ZC}_k(V) = \mathcal{C}_k(V) \cap \mathrm{Ker}\delta$ and $\mathcal{BC}_k(V) = \mathcal{C}_k(V) \cap \mathrm{Im}\delta$.

Note that given $g \in \mathcal{C}_1(V)$ and $h \in \mathcal{C}_2(V)$, for any $s,u,t \in [0,T]$, we have

$$(\delta g)_{st} = g_t - g_s, \quad \text{ and } \quad (\delta h)_{sut} = h_{st} - h_{su} - h_{ut}. \tag{3.2}$$

Furthermore, it can be checked that $\mathcal{ZC}_k(V) = \mathcal{BC}_k(V)$ for any $k \geq 1$. Moreover, the following property holds:

Lemma 3.1 *Let $k \geq 1$ and $h \in \mathcal{ZC}_{k+1}(V)$. Then there exists a (non unique) $f \in \mathcal{C}_k(V)$ such that $h = \delta f$.*

Observe that Lemma 3.1 yields that all the elements $h \in \mathcal{C}_2(V)$ such that $\delta h = 0$ can be written as $h = \delta f$ for some (non unique) $f \in \mathcal{C}_1(V)$.

Basically, we will use k-increments with $k \leq 2$. We measure the size of these increments by Hölder norms defined in the following way: for $0 \leq a_1 < a_2 \leq T$ and $f \in \mathcal{C}_2([a_1,a_2];V)$, let

$$\|f\|_{\mu,[a_1,a_2]} = \sup_{r,t\in[a_1,a_2]} \frac{|f_{rt}|}{|t-r|^{\mu}},$$

and

$$\mathcal{C}_2^{\mu}([a_1,a_2];V) = \left\{ f \in \mathcal{C}_2(V);\ \|f\|_{\mu,[a_1,a_2]} < \infty \right\}.$$

Notice that the usual Hölder spaces $\mathcal{C}_1^{\mu}([a_1,a_2];V)$ will be determined in the following way: for a continuous function $g \in \mathcal{C}_1([a_1,a_2];V)$, we set

$$\|g\|_{\mu,[a_1,a_2]} = \|\delta g\|_{\mu,[a_1,a_2]}. \tag{3.3}$$

We will say that $g \in \mathcal{C}_1^{\mu}([a_1,a_2];V)$ if $\|g\|_{\mu,[a_1,a_2]}$ is finite.

For $h \in \mathcal{C}_3([a_1,a_2];V)$ set now

$$\begin{aligned} \|h\|_{\gamma,\rho,[a_1,a_2]} &= \sup_{s,u,t\in[a_1,a_2]} \frac{|h_{sut}|}{|u-s|^{\gamma}|t-u|^{\rho}}, \\ \|h\|_{\mu,[a_1,a_2]} &= \inf\left\{ \sum_i \|h_i\|_{\rho_i,\mu-\rho_i,[a_1,a_2]};\ h = \sum_i h_i,\ 0 < \rho_i < \mu \right\}, \end{aligned} \tag{3.4}$$

where the last infimum is taken over all sequences $\{h_i \in \mathcal{C}_3(V)\}$ such that $h = \sum_i h_i$ and for all choices of the numbers $\rho_i \in (0,\mu)$. Then $\|\cdot\|_{\mu}$ is a norm on $\mathcal{C}_3([a_1,a_2];V)$, and we set

$$\mathcal{C}_3^{\mu}([a_1,a_2];V) := \left\{ h \in \mathcal{C}_3([a_1,a_2];V);\ \|h\|_{\mu} < \infty \right\}.$$

Consider $\mathcal{C}_3^{1+}([a_1,a_2];V) = \cup_{\mu>1}\mathcal{C}_3^{\mu}([a_1,a_2];V)$. Notice that the same kind of norms can be considered on the spaces $\mathcal{ZC}_3([a_1,a_2];V)$, leading to the definition of the spaces $\mathcal{ZC}_3^{\mu}([a_1,a_2];V)$ and $\mathcal{ZC}_3^{1+}([a_1,a_2];V)$.

The basic point in this approach to pathwise integration of irregular processes is that, under smoothness conditions, the operator δ can be inverted. This inverse, called Λ, is defined in the following proposition, taken from [14] and whose proof can be found in [11].

Proposition 3.2 *Let $0 \le a_1 < a_2 \le T$. Then there exists a unique linear map $\Lambda : \mathcal{ZC}_3^{1+}([a_1,a_2];V) \to \mathcal{C}_2^{1+}([a_1,a_2];V)$ such that*

$$\delta\Lambda = Id_{\mathcal{ZC}_3^{1+}([a_1,a_2];V)}.$$

In other words, for any $h \in \mathcal{C}_3^{1+}([a_1,a_2];V)$ such that $\delta h = 0$ there exists a unique $g = \Lambda(h) \in \mathcal{C}_2^{1+}([a_1,a_2];V)$ such that $\delta g = h$. Furthermore, for any $\mu > 1$, the map Λ is continuous from $\mathcal{ZC}_3^{\mu}([a_1,a_2];V)$ to $\mathcal{C}_2^{\mu}([a_1,a_2];V)$ and we have

$$\|\Lambda h\|_{\mu,[a_1,a_2]} \le \frac{1}{2^\mu - 2}\|h\|_{\mu,[a_1,a_2]}, \qquad h \in \mathcal{ZC}_3^{\mu}([a_1,a_2];V). \tag{3.5}$$

3.2 Young integration

We will consider now the particular case where $V = \mathbb{R}^n$, for an arbitrary $n \ge 1$.

Using the tools introduced in the previous subsection, here we will present a generalized integral $\int_s^t f_u dg_u$ for $f \in C_1^{\kappa}([0,T];\mathbb{R}^{n\times d})$ and $g \in \mathcal{C}_1^{\gamma}([0,T];\mathbb{R}^d)$. Following the notations introduced in [14, 7], we will sometimes write $\mathcal{J}_{st}(f\,dg)$ instead of $\int_s^t f_u dg_u$.

Let us consider first two smooth functions f and g defined on $[0,T]$. One can write,

$$\mathcal{J}_{st}(f\,dg) \equiv \int_s^t f_u\,dg_u = f_s(\delta g)_{st} + \int_s^t (\delta f)_{su}\,dg_u = f_s(\delta g)_{st} + \mathcal{J}_{st}(\delta f\,dg). \tag{3.6}$$

Let us study the term $\mathcal{J}(\delta f\,dg)$. It is easily seen that, for $s,u,t \in [0,T]$,

$$h_{sut} \equiv [\delta\left(\mathcal{J}(\delta f\,dg)\right)]_{sut} = (\delta f)_{su}(\delta g)_{ut}.$$

The increment h is an element of $\mathcal{C}_3(\mathbb{R}^n)$ satisfying $\delta h = 0$. Let us estimate now the regularity of h: if $f \in C_1^{\kappa}([0,T];\mathbb{R}^{n\times d})$ and $g \in \mathcal{C}_1^{\gamma}([0,T];\mathbb{R}^d)$, from (3.4), it is easily checked that $h \in \mathcal{C}_3^{\gamma+\kappa}(\mathbb{R}^n)$. Hence $h \in \mathcal{ZC}_3^{\gamma+\kappa}(\mathbb{R}^n)$, and if $\kappa + \gamma > 1$ (which is the case if f and g are regular), Proposition 3.2 implies that $\mathcal{J}(\delta f\,dg)$ can be written as

$$\mathcal{J}(\delta f\,dg) = \Lambda(h) = \Lambda\left(\delta f\,\delta g\right),$$

and thus, plugging this identity into (3.6), we get:

$$\mathcal{J}_{st}(f\,dg) = f_s(\delta g)_{st} + \Lambda_{st}\left(\delta f\,\delta g\right). \tag{3.7}$$

Let us state an extension of Theorem 2.5 of [14] where it is extended the notion of integral whenever $f \in C_1^\kappa([0,T];\mathbb{R}^{n\times d})$ and $g \in \mathcal{C}_1^\gamma([0,T];\mathbb{R}^d)$.

Theorem 3.3 *Let $f \in C_1^k([0,T];\mathbb{R}^{n\times d})$ and $g \in C_1^\gamma([0,T];\mathbb{R}^d)$ with $k+\gamma > 1$. Set*

$$\int_s^t f dg = f_s(\delta g)_{st} + \Lambda_{st}(\delta f \delta g).$$

Then:

1. *Whenever f and g are smooth functions, $\int_s^t f dg$ coincides with the usual Riemann integral.*
2. *$\int_s^t f dg$ coincides with the Young integral as defined in [19].*
3. *For any $\beta \in [0,1)$ such that $1 < \gamma + k(1-\beta) = \mu_\beta$, the generalized integral satisfies*

$$|\int_s^t f dg| \le \|f\|_{\infty,[s,t]} \|g\|_\gamma |t-s|^\gamma + c_{\gamma,k,\beta} \|f\|_{\infty,[s,t]}^\beta \|f\|_{k,[s,t]}^{1-\beta} \|g\|_\gamma |t-s|^{\mu_\beta}, \tag{3.8}$$

where $c_{\gamma,k,\beta} = 2^\beta(2^{\mu_\beta}-1)^{-1}$.

Proof:

The proof of the original Theorem has been presented in [11] (see also [12], [14]). The first two statements of our Theorem are exactly the same that those in Theorem 2.5 in [14], so we refer the reader to this reference for their proof.

The last statement is a generalization of the one presented in Theorem 2.5 of [14], where it is only considered the case $\beta = 0$. The proof for $\beta > 0$ can be obtained easily putting together the following inequality

$$\|f\|_{k(1-\beta),[s,t]} \le 2^\beta \|f\|_{\infty,[s,t]}^\beta \|f\|_{k,[s,t]}^{1-\beta}$$

and the inequality given in Theorem 2.5 of [14]

$$|\int_s^t f dg| \le \|f\|_{\infty,[s,t]} \|g\|_\gamma |t-s|^\gamma + c_{\gamma,k(1-\beta)} \|f\|_{k(1-\beta),[s,t]} \|g\|_\gamma |t-s|^{\gamma+k(1-\beta)},$$

where $c_{\gamma,k(1-\beta)} = (2^{\gamma+k(1-\beta)}-1)^{-1}$.

□

3.3 Skorohod mapping

We recall here from [8] a well-known result for the Skorohod mapping.

Lemma 3.4 *For each path $z \in \mathcal{C}(\mathbb{R}_+,\mathbb{R}^d)$, there exists a unique solution (x,y) to the Skorokhod problem for z. Thus there exists a pair of functions $(\phi,\varphi) : \mathcal{C}_+(\mathbb{R}_+,\mathbb{R}^d) \to \mathcal{C}_+(\mathbb{R}_+,\mathbb{R}^{2d})$ defined by $(\phi(z),\varphi(z)) = (x,y)$. The pair (ϕ,φ) satisfies the following:*

There exists a constant $K_l > 0$ such that for any $z_1, z_2 \in \mathcal{C}_+(\mathbb{R}_+, \mathbb{R}^d)$ we have for each $t \geq 0$,

$$\begin{aligned} \|\phi(z_1) - \phi(z_2)\|_{\infty,[0,t]} &\leq K_l \|z_1 - z_2\|_{\infty,[0,t]}\,, \\ \|\varphi(z_1) - \varphi(z_2)\|_{\infty,[0,t]} &\leq K_l |z_1 - z_2\|_{\infty,[0,t]}\,. \end{aligned}$$

In our paper we will use that the $\lambda-$Hölder norm of the regulator term y is bounded by that of z, as proven in the following easy lemma.

Lemma 3.5 *Consider $z \in \mathcal{C}(\mathbb{R}_+, \mathbb{R}^d)$, such that $\|z\|_{\lambda,[0,T]} < \infty$. Then for any $0 \leq s \leq t \leq T$*

$$\|y\|_{\lambda,[s,t]} \leq C_d \|z\|_{\lambda,[s,t]}.$$

Proof:
Take u, v such that $s \leq u < v \leq t$. Fixed a component i, we wish to study

$$\frac{|y^i_v - y^i_u|}{(v-u)^\lambda}.$$

When $y^i_u = y^i_v$, this is clearly zero. On the other hand, when $y^i_v > y^i_u$, let us define

$$\begin{aligned} u^* &:= \sup\{u' \geq u; y^i_{u'} = y^i_u\}, \\ v^* &:= \inf\{v' \leq v; y^i_{v'} = y^i_v\}. \end{aligned}$$

Then, $u \leq u^* < v^* \leq v$ and $y^i_u = y^i_{u^*}, y^i_v = y^i_{v^*}$. So

$$\frac{|y^i_v - y^i_u|}{(v-u)^\lambda} \leq \frac{|y^i_{v^*} - y^i_{u^*}|}{(v^* - u^*)^\lambda} = \frac{|z^i_{v^*} - z^i_{u^*}|}{(v^* - u^*)^\lambda}$$

where the last equality follows from the fact that y^i and z^i coincides whenever y^i is not constant.

Then, note that

$$\sup_{s \leq u < v \leq t} \frac{|y^i_v - y^i_u|}{(v-u)^\lambda} \leq \sup_{s \leq u^* < v^* \leq t} \frac{|z^i_{v^*} - z^i_{u^*}|}{(v^* - u^*)^\lambda} \leq \|z\|_{\lambda,[s,t]}.$$

Finally, we get that

$$\|y\|_{\lambda,[s,t]} \leq \Big(\sum_{i=1}^d \big(\sup_{s \leq u < v \leq t} \frac{|y^i_v - y^i_u|}{(v-u)^\lambda} \big)^2 \Big)^{\frac{1}{2}} \leq d^{\frac{1}{2}} \|z\|_{\lambda,[s,t]}.$$

$\square$

Remark 3.6 *It is possible to prove that a similar estimate does not hold for the difference of two regulator terms, result that we would need in order to prove a*

uniqueness theorem in the Hölder norm framework. Indeed, let $0 < t_1 < t_2 < t$, $\lambda \in (0,1)$, *and take* $z^1, z^2 \in C^\lambda([0,t])$ *defined as*

$$z^1(s) = [(t_2 - s)/(t_2 - t_1) - 1]\mathbf{1}_{(t_1,t_2]}(s) - \mathbf{1}_{(t_2,t]}(s)$$

$$z^2(s) = s/t_1\mathbf{1}_{[0,t_1]}(s) + (t_2 - s)/(t_2 - t_1)\mathbf{1}_{(t_1,t_2]}(s)$$

(note that $z^1(0) = z^2(0)$*). It is easy to see that* $y^1(s) = [1 - (t_2 - s)/(t_2 - t_1)]\mathbf{1}_{(t_1,t_2]}(s) + \mathbf{1}_{(t_2,t]}(s)$, *while* $y^2(s) \equiv 0$. *We get then*

$$\|y^2 - y^1\|_{\lambda,[0,t]} = \|y^1\|_{\lambda,[0,t]} = \frac{1}{(t_2 - t_1)^\lambda}$$

while

$$\|z^2 - z^1\|_{\lambda,[0,t]} = \frac{1}{(t_1)^\lambda}$$

Taking t_1 *fixed and* $t_2 - t_1$ *small, we prove that in general the* $\lambda-$*Hölder norm of the difference of two regulator terms cannot be bounded by the* $\lambda-$*Hölder norm of the difference of* z^1 *and* z^2.

4 Deterministic integral equations

In this section we will prove all the deterministic results.

Consider the deterministic differential equation on $\mathbb{R}^d$

$$x(t) = x(0) + \int_0^t b(s, x(s))ds + \int_0^t \sigma(s, x(s))dg_s + y(t), \quad t \in (0, T], \tag{4.1}$$

where for each $i = 1, \ldots, d$

$$y^i(t) = \max_{s\in[0,t]} \left(z^i(s)\right)^-, \qquad t \in [0, T],$$

and

$$z(t) = x(0) + \int_0^t b(s, x(s))ds + \int_0^t \sigma(s, x(s))dg_s, \qquad t \in [0, T].$$

We will assume that the driving noise g belongs to $C^\gamma([0, T]; \mathbb{R}^m)$ with $\gamma > \frac{1}{2}$. Then, the integral with respect to g has to be interpreted in the Young sense and we will find a solution x in the space $C^\lambda([0, T]; \mathbb{R}^d)$ with $\lambda \in (\frac{1}{2}, \gamma)$.

The result of existence reads as follows.

Theorem 4.1 *Assume that* σ *and* b *satisfy hypothesis* **(H1)** *and* **(H2)**, *respectively, with* $\nu \geq \gamma$. *Set* $\lambda \in (\frac{1}{2}, \gamma)$. *Then equation (4.1) has a solution* $x \in C^\lambda([0, T]; \mathbb{R}^d_+)$.

Proof:

To prove that equation (4.1) admits a solution on $[0,T]$, we shall prove first that it has a solution on $[0,T_1]$ for T_1 small enough (T_1 will be defined later). Then we will extend the solution to $[0,T]$ using an induction argument to extend the result from $[0,nT_1]$ to $[0,(n+1)T_1]$.

STEP 1: Study on $[0,T_1]$.

Let us consider

$$x^{(1)}(t) = z^{(1)}(t) = x(0); y^{(1)}(t) = 0 \qquad t \in [0,T], \tag{4.2}$$

and for all $n > 1$

$$x^{(n+1)}(t) = x(0)+\int_0^t b(s,x^{(n)}(s))ds+\int_0^t \sigma(s,x^{(n)}(s))dg_s+y^{(n)}(t), \qquad t \in [0,T], \tag{4.3}$$

where for each $i = 1,\dots,d$

$$y^{(n),i}(t) = \max_{s\in[0,t]} \left(z^{(n),i}(s)\right)^-, \qquad t \in [0,T],$$

with

$$z^{(n)}(t) = x(0) + \int_0^t b(s,x^{(n)}(s))ds + \int_0^t \sigma(s,x^{(n)}(s))dg_s, \qquad t \in [0,T].$$

Step 1.1: Properties of the functions $x^{(n)}$

It follows that $x^{(n)} \in C^\lambda([0,T_1],\mathbb{R}^d_+)$ for all $n \geq 1$. Indeed, from Lemma 3.5, we have that

$$\|x^{(n+1)}\|_{\lambda,[0,T_1]} \leq \|z^{(n)}\|_{\lambda,[0,T_1]} + \|y^{(n)}\|_{\lambda,[0,T_1]} \leq C_d \|z^{(n)}\|_{\lambda,[0,T_1]}.$$

Using Theorem 3.3 and the hypothesis on the coefficients

$$\begin{aligned} &|\int_s^t b(u,x^{(n)}(u))du + \int_s^t \sigma(u,x^{(n)}(u))dg_u| \\ &\quad \leq \|b\|_\infty |t-s| + \|\sigma\|_\infty \|g\|_\gamma |t-s|^\gamma \\ &\qquad + c_{\gamma,\lambda}\|g\|_\gamma \|\sigma(.,x^{(n)}(.))\|_{\lambda,[s,t]} |t-s|^{\gamma+\lambda}. \end{aligned}$$

Using that

$$\|\sigma(.,x^{(n)}(.))\|_{\lambda,[s,t]} \leq K_0 \left(\|x^{(n)}\|_{\lambda,[s,t]} + |t-s|^{\nu-\lambda} \right),$$

we can write that

$$\|x^{(n+1)}\|_{\lambda,[s,t]} \leq h(t-s) + M_1 \|x^{(n)}\|_{\lambda,[s,t]} |t-s|^\gamma, \tag{4.4}$$

where $h(t) = C_d(\|b\|_\infty t^{1-\lambda} + \|\sigma\|_\infty \|g\|_\gamma t^{\gamma-\lambda} + c_{\gamma,\lambda} K_0 \|g\|_\gamma t^{\gamma-\lambda+\nu})$, $M_1 = C_d c_{\gamma,\lambda} K_0 \|g\|_\gamma$. Repeating iterativily inequality (4.4) with $s = 0$ we get that

$$\|x^{(n+1)}\|_{\lambda,[0,t]} \leq h(t) \left(\sum_{i=0}^{n-1} (M_1 t^\gamma)^i + (M_1 t^\gamma)^n \|x^{(1)}\|_{\lambda,[0,t]} \right) = h(t) \sum_{i=0}^{n-1} (M_1 t^\gamma)^i. \tag{4.5}$$

So, choosing T_1 such that $T_1 < (1/M_1)^{1/\gamma}$,

$$\sup_{n \geq 1} \|x^{(n)}\|_{\lambda,[0,T_1]} := K_1 < \infty. \tag{4.6}$$

Since $x^{(n)}(0) = x(0)$ for all n, it follows easily that

$$\sup_{n \geq 1} \|x^{(n)}\|_{\infty,[0,T_1]} := K_2 < \infty. \tag{4.7}$$

Step 1.2: Definition of the solution

The sequence of functions $x^{(n)}$ is equicontinuous and bounded in $C([0,T_1];\mathbb{R}^d)$. Therefore there exists a subsequence $\{x^{(n_j)}\}_{j\geq 1}$ that converges uniformly to a function $x \in C([0,T_1];\mathbb{R}^d)$.

Moreover x belongs to $C^\lambda([0,T_1],\mathbb{R}^d_+)$. Indeed, fixed $\varepsilon > 0$ let us choose n_j such that $\|x - x^{n_j}\|_{\infty,[0,T_1]} \leq \varepsilon$. Then, for all $s,t \in [0,T_1]$

$$\begin{aligned} |x(t) - x(s)| &\leq |x(t) - x^{(n_j)}(t)| + |x^{(n_j)}(t) - x^{(n_j)}(s)| + |x^{(n_j)}(s) - x(s)| \\ &\leq 2\varepsilon + K_1 |t-s|^\lambda. \end{aligned}$$

Since this inequality is true for all $\varepsilon > 0$, we get that $x \in C^\lambda([0,T_1];\mathbb{R}^d_+)$ and, for any $t \in [0,T_1]$, the Young integral

$$\int_0^t \sigma(s,x(s))dg_s$$

is well-defined.

Step 1.3: x satisfies (4.1) on $[0,T_1]$.

Since $\{x^{(n_j)}\}_{j\geq 1}$ converges uniformly to x and b is Lipschitz in space, we get that

$$\lim_{n_j \to \infty} \| \int_0^\cdot b(s,x^{(n_j)}(s))ds - \int_0^\cdot b(s,x(s))ds \|_{\infty,[0,T_1]} = 0. \tag{4.8}$$

On the other hand, using the hypothesis on the coefficients and Theorem 3.3, for any $t \in [0,T_1]$:

$$\begin{aligned} & \left| \int_0^t \left(\sigma(s,x(s)) - \sigma(s,x^{(n_j)}(s)) \right) dg_s \right| \\ \leq \;& K_0 \|x^{(n)} - x^{(n-1)}\|_{\infty,[0,T_1]} \|g\|_\gamma T_1^\gamma \\ & \quad + K_0^\beta c_{\gamma,\lambda,\beta} \|x - x^{(n_j)}\|^\beta_{\infty,[0,T_1]} \|\sigma(.,x(.)) - \sigma(.,x^{(n_j)}(.))\|^{1-\beta}_{\lambda,[0,T_1]} \|g\|_\gamma T_1^{\mu_\beta}. \end{aligned}$$

Since $\|\sigma(.,x(.)) - \sigma(.,x^{(n_j)}(.))\|_{\lambda,[0,T_1]} < \infty$,

$$\lim_{n_j\to\infty} \| \int_0^{\cdot} \sigma(s,x(s))dg_s - \int_0^{\cdot} \sigma(s,x^{(n_j)}(s))dg_s \|_{\infty,[0,T_1]} = 0. \tag{4.9}$$

Finally, for each $i = 1,\dots,d$, set

$$y^i(t) = \max_{s\in[0,t]} \left(z^i(s)\right)^-, \qquad t \in [0,T_1],$$

where

$$z(t) = x_0 + \int_0^t b(s,x(s))ds + \int_0^t \sigma(s,x(s))dg_s, \qquad t \in [0,T_1].$$

Using Lemma 3.4 we have

$$\sup_{t\in[0,T_1]} \left| y(t) - y^{(n_j)}(t) \right| \le K_l \sup_{t\in[0,T_1]} \left| z(t) - z^{(n_j)}(t) \right|.$$

From (4.8) and (4.9) we obtain now that

$$\lim_{n_j\to\infty} \|y - y^{(n_j)}\|_{\infty,[0,T_1]} = 0. \tag{4.10}$$

Letting n_j to infinity in (4.3) and using (4.8) ,(4.9) and (4.10), we get that x satisfies (4.1).

STEP 2.: We will assume that we have defined the solution on $[0, NT_1]$. We will show first the extension to $[NT_1,(N+1)T_1]$ (assuming $(N+1)T_1 \le T$).

Let x be a solution defined in $[0, NT_1]$. Then, let us define

$$x^{(1)}(t) := x(t)\mathbf{1}_{[0,NT_1]}(t) + x(T_1)\mathbf{1}_{(NT_1,T]}(t),$$
$$z^{(1)}(t) := z(t)\mathbf{1}_{[0,NT_1]}(t) + z(T_1)\mathbf{1}_{(NT_1,T]}(t).$$

Moreover, for all $n > 1$

$$x^{(n+1)}(t) = x(t), \qquad t \in [0, NT_1],$$
$$x^{(n+1)}(t) = z(NT_1) + \int_{NT_1}^t b(s,x^{(n)}(s))ds + \int_{NT_1}^t \sigma(s,x^{(n)}(s))dg_s + y^{(n)}(t),$$
$$t \in (NT_1,T],$$

where for each $i = 1,\dots,d$

$$y^{(n),i}(t) = \max_{s\in[0,t]} \left(z^{(n),i}(s)\right)^-, \qquad t \in [NT_1,T],$$

with

$$z^{(n)}(t) = z(t), \qquad t \in [0,T_1],$$
$$z^{(n)}(t) = z(NT_1) + \int_{NT_1}^t b(s,x^{(n)}(s))ds + \int_{NT_1}^t \sigma(s,x^{(n)}(s))dg_s, ,$$
$$t \in (NT_1,T].$$

Note that

$$y^{(1)}(t) := y(t)\mathbf{1}_{[0,NT_1]}(t) + y(T_1)\mathbf{1}_{(NT_1,T]}(t).$$

Repeating the same computations given in *Step 1.1* we get that

$$\|x^{(n+1)}\|_{\lambda,[NT_1,t]} \leq h(t-NT_1)\sum_{i=0}^{n-1}(M_1(t-NT_1)^{\gamma})^i, \tag{4.11}$$

where $h(t)$ and M_1 are the same that appear in (4.5). Using the same ideas employed in *Step 1.1*, we obtain that

$$\sup_{n\geq 1}\|x^{(n)}\|_{\lambda,[NT_1,(N+1)T_1]} := K_1 < \infty,$$

and

$$\sup_{n\geq 1}\|x^{(n)}\|_{\infty,[NT_1,(N+1)T_1]} := K_2 < \infty.$$

Following now the method used in *Step 1.2* and *Step 1.3*, there exists a subsequence $\{x^{(n_j)}\}_{j\geq 1}$ that converges uniformly to a function $x^{[2]} \in C([NT_1,(N+1)T_1];\mathbb{R}^d)$. Moreover $x^{[2]}$ belongs to $C^{\lambda}([NT_1,(N+1)T_1],\mathbb{R}^d_+)$ and $x^{[2]}$ satisfies (4.1) in $[NT_1,(N+1)T_1]$.

Set now:

$$x(t) = x(t)\mathbf{1}_{[0,NT_1]}(t) + x^{[2]}(t)\mathbf{1}_{(NT_1,(N+1)T_1]}(t).$$

Clearly, x belongs to $C^{\lambda}([0,(N+1)T_1],\mathbb{R}^d)$ and x satisfies (4.1) in $[0,(N+1)T_1]$. $\square$

Remark 4.2 *The study of the uniqueness of the solution is an open problem, due to the fact that we are not able to bound the Hölder norm of the difference of regulator terms with the same norm of the difference of the reflected terms. We can only get the uniqueness in a small time interval and when σ does not depend on time. Assuming that $\sigma:\mathbb{R}\to\mathbb{R}^m$ is bounded and Lipschitz continuous, b satisfy hypothesis* **(H2)**, *$\nu\geq\gamma$ and $\lambda_0\in(\frac{1}{2},\gamma)$, then there exists $\varepsilon>0$ such that equation (4.1) has a unique solution on $[0,\varepsilon]$ and $x\in C^{\lambda_0}([0,\varepsilon];\mathbb{R}^d_+)$.*

Indeed, fix x a solution of our equation in $[0,T]$. Since $x^i(0)>0$ for any i, there exists $\varepsilon>0$ such that $x^i(t)>0$ for any i and $t\in[0,\varepsilon]$. Let x_1 be a second solution of our equation and let $\delta:=\sup\{t\geq 0: x_1^i(t)>0, i=1,\ldots,k\}$. Clearly $\delta>0$ and assume, w.l.g. that $\delta\leq\varepsilon$. On $[0,\delta)$, both solutions x and x_1 are nonnegative and $y(s)=y_1(s)=0$ for all $s\in[0,\delta)$. For any $t_0\in[0,\delta)$

$$\begin{aligned}\|x-x_1\|_{\lambda,[0,t_0]} &\leq \|\int_0^{\cdot}(b(u,x^a(u))-b(u,x^b(u)))du\|_{\lambda,[0,t_0]}\\ &\quad+\|\int_0^{\cdot}(\sigma(x(u))-\sigma(x_1(u)))dg_u\|_{\lambda,[0,t_0]}.\end{aligned} \tag{4.12}$$

On one hand

$$\begin{aligned}
&\| \int_0^{\cdot} (b(u,x(u)) - b(u,x_1(u)))du\|_{\lambda,[0,t_0]} \\
&\quad = \sup_{0\le s<t\le t_0} \frac{|\int_s^t (b(u,x(u)) - b(u,x_1(u)))du|}{|t-s|^{\lambda}} \\
&\quad \le K_0\|x-x_1\|_{\infty,[0,t_0]} t_0^{1-\lambda} \le K_0\|x-x_1\|_{\lambda,[0,t_0]} t_0. \qquad (4.13)
\end{aligned}$$

On the other hand, using Theorem 3.3

$$\begin{aligned}
&\| \int_0^{\cdot} (\sigma(x(u)) - \sigma(x_1(u)))dg_u\|_{\lambda,[0,t_0]} \\
&\quad \le K_0\|x-x_1\|_{\infty,[0,t_0]}\|g\|_{\gamma} t_0^{\gamma-\lambda} + c_{\gamma,\lambda}\|x-x_1\|_{\lambda,[0,t_0]}\|g\|_{\gamma} t_0^{\gamma} \\
&\quad \le (K_0 + c_{\gamma,\lambda})\|x-x_1\|_{\lambda,[0,t_0]}\|g\|_{\gamma} t_0^{\gamma}. \qquad (4.14)
\end{aligned}$$

Putting (4.13) and (4.14) in (4.12) we get that

$$\|x^a - x^b\|_{\lambda,[0,t_0]} \le \Big(K_0 t_0 + (K_0 + c_{\gamma,\lambda})\|g\|_{\gamma} t_0^{\gamma}\Big)\|x-x_1\|_{\lambda,[0,t_0]}.$$

Choosing t_0 small enough it follows that $\|x-x_1\|_{\lambda,[0,t_0]} = 0$. Since $x(0) = x_1(0)$ it follows that $x = x_1$ on $[0,t_0]$. Since we can repeat this arguments on $[t_0, 2t_0]$ and further on, we get that $x = x_1$ on $[0,\delta]$.

From the continuity of our solutions we have that $x(\delta) = x_1(\delta) = 0$, $y(\delta) = y_1(\delta) = 0$ and $\delta = \varepsilon$.

We will finish the study of the deterministic case obtaining a bound for the Hölder norm of the solutions.

Theorem 4.3 *Assume that σ and b satisfy hypothesis* **(H1)** *and* **(H2)**, *respectively, with $\nu \ge \gamma$ and set $\lambda \in (\frac{1}{2}, \gamma)$. Given x a solution of equation (4.1), it holds that $\|x\|_{\lambda,[0,T]} \le M_2 + M_3\|g\|_{\gamma}^{\frac{1}{\gamma}}$, where M_2 and M_3 are positive constants not depending on g.*

Proof:
Using that b and σ are bounded, Lemma 3.5, Theorem 3.3 and that

$$\|\sigma(.,x(.))\|_{\lambda,[s,t]} \le K_0|t-s|^{\nu-\lambda} + K_0\|x\|_{\lambda,[s,t]},$$

we get that for any $s \le t$

$$\begin{aligned}
\|x\|_{\lambda,[s,t]} &\le \|z\|_{\lambda,[s,t]} + \|y\|_{\lambda,[s,t]} \le (C_d+1)\|z\|_{\lambda,[s,t]} \\
&\le (C_d+1)\left(\|b\|_{\infty}|t-s|^{1-\lambda} + \|\sigma\|_{\infty}\|g\|_{\gamma}|t-s|^{\gamma-\lambda}\right. \\
&\qquad \left. + c_{\gamma,\lambda}\|\sigma(.,x(.))\|_{\lambda,[s,t]}\|g\|_{\gamma}|t-s|^{\gamma}\right) \\
&\le (C_d+1)\left(\|b\|_{\infty}|t-s|^{1-\lambda} + (\|\sigma\|_{\infty} + T^{\nu}K_0 c_{\gamma,\lambda})\|g\|_{\gamma}|t-s|^{\gamma-\lambda}\right. \\
&\qquad \left. + K_0 c_{\gamma,\lambda}\|x\|_{\lambda,[s,t]}\|g\|_{\gamma}|t-s|^{\gamma}\right).
\end{aligned}$$

Set $M_{d,\gamma,\lambda} := (C_d + 1)K_0 c_{\gamma,\lambda}$. So, for any $0 \le s < t \le T$ such that

$$M_{d,\gamma,\lambda}\|g\|_\gamma |t-s|^\gamma \le \frac{1}{2}, \tag{4.15}$$

we get that

$$\|x\|_{\lambda,[s,t]} \le 2(C_d+1)\left(\|b\|_\infty |t-s|^{1-\lambda} + (\|\sigma\|_\infty + T^\nu K_0 c_{\gamma,\lambda})\|g\|_\gamma |t-s|^{\gamma-\lambda}\right) \tag{4.16}$$

Note that given any $0 \le s < t \le T$ that do not satisfy (4.15), we can choose $t_0 = s < t_1 < \ldots < t_n = t$ such that for all $i \in 1, \ldots, n$

$$M_{d,\gamma,\lambda}\|g\|_\gamma |t_i - t_{i-1}|^\gamma \le \frac{1}{2}$$

with

$$n \le 2|t-s| \left(2M_{d,\gamma,\lambda}\|g\|_\gamma\right)^{\frac{1}{\gamma}}.$$

Then, using (4.16), we have that

$$\begin{aligned}
\frac{|x(t)-x(s)|}{|t-s|^\lambda} &\le \sum_{i=1}^n \frac{|x(t_i)-x(t_{i-1})|}{|t_i-t_{i-1}|^\lambda}\frac{|t_i-t_{i-1}|^\lambda}{|t-s|^\lambda} \\
&\le \sum_{i=1}^n \|x\|_{\lambda,[t_{i-1},t_i]}\frac{|t_i-t_{i-1}|^\lambda}{|t-s|^\lambda} \\
&\le \sum_{i=1}^n \frac{2(C_d+1)\left(\|b\|_\infty|t_i-t_{i-1}| + (\|\sigma\|_\infty + T^\nu K_0 c_{\gamma,\lambda})\|g\|_\gamma |t_i-t_{i-1}|^\gamma\right)}{|t-s|^\lambda} \\
&\le 2(C_d+1)\|b\|_\infty T^{1-\lambda} \\
&\quad + \sum_{i=1}^n \frac{2(C_d+1)\left((\|\sigma\|_\infty + T^\nu K_0 c_{\gamma,\lambda})\|g\|_\gamma |t_i-t_{i-1}|^\gamma\right)}{|t-s|^\lambda} \\
&\le 2(C_d+1)\|b\|_\infty T^{1-\lambda} \\
&\quad + \left(2|t-s|\left(2M_{d,\gamma,\lambda}\|g\|_\gamma\right)^{\frac{1}{\gamma}}\right)\frac{2(C_d+1)\left((\|\sigma\|_\infty + T^\nu K_0 c_{\gamma,\lambda})\|g\|_\gamma\right)}{|t-s|^\lambda(2M_{d,\gamma,\lambda}\|g\|_\gamma)} \\
&\le 2(C_d+1)\|b\|_\infty T^{1-\lambda} \\
&\quad + T^{1-\lambda}2^{1+\frac{1}{\gamma}} M_{d,\gamma,\lambda}^{\frac{1}{\gamma}-1}(C_d+1)\left(\|\sigma\|_\infty + T^\nu K_0 c_{\gamma,\lambda}\right)\|g\|_\gamma^{\frac{1}{\gamma}}.
\end{aligned}$$

From this last inequality, it follows easily that $\|x\|_{\lambda,[0,T]} \le M_2 + M_3\|g\|_\gamma^{\frac{1}{\gamma}}$.
□

5 Stochastic integral equations

In this section we apply the deterministic results to prove the main theorems of this paper.

The following Lemma, taken from [17] (see Lemma 7.4) is basic in order to extend the deterministic results to the stochastic ones.

Lemma 5.1 *Let $\{W_t^H; t \geq 0\}$ be a fractional Brownian motion of Hurst parameter $H \in (0,1)$. Then for every $0 < \varepsilon < H$ and $T > 0$ there exists a positive random variable $\eta_{\varepsilon,T}$ such that $E(|\eta_{\varepsilon,T}|^p) < \infty$ for all $p \in [1,\infty)$ and for all $s,t \in [0,T]$*

$$|W_t^H - W_s^H| \leq \eta_{\varepsilon,T}|t-s|^{H-\varepsilon}, \qquad \text{a.s.}$$

The stochastic integral appearing throughout this paper $\int_0^T u(s)dW_s^H$ is a Young integral. This integral exists if the process $u(s)$ has Hölder continuous trajectories of order larger than λ such that $H + \lambda > 1$.

On the other hand, notice that from lemma 5.1, for any $\gamma < H$ it holds that

$$E(\|W^H\|_\gamma^p) < \infty,$$

for all $p \in [1,\infty)$.

Choosing γ and λ, such that $\gamma = H - \varepsilon_1$ with $\gamma > \frac{1}{2}$ and $\lambda \in (\frac{1}{2}, \gamma)$, Theorems 4.1 and 4.3 follows almost surely. Our stochastic theorems of existence and boundedness of the moments hold then clearly.

Acknowledgements

We would like to thank A. Deya for his comments that have helped us to simplify the proof of Theorem 3.3.

This work was partially supported by COFIN 2008-YYYBE4-002 of MIUR (Marco Ferrante) and DGES Grant MTM09-07203 (Carles Rovira).

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